On unsteady laminar boundary layers

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A transformation is introduced which, for a class of outer pressure distributions, reduces the unsteady incompressible laminar boundary-layer equations in two dimensions to an equation in which the time does not appear explicitly. A formally exact solution of the resulting equation is then presented in the form of a series and it is shown that the solution can be expressed in terms of universal functions.

1. Introduction

Early attempts to discuss unsteady laminar boundary-layer flows were mainly restricted to early phases of a motion starting from rest and to oscillatory motions without a mean flow. Very little attention was paid to the subject because it was felt that boundary-layer growth took place in such a short time that the flow may be considered steady. It is clear that such considerations do not apply if one considers the problem of a vehicle moving with variable speed over its entire trajectory, and a detailed investigation of unsteady flows is required if reliable information concerning skin friction and heat transfer is desired.

Recently Lighthill (1954) considered the influence of free-stream fluctuations on skin friction and heat transfer, and Yang (1958) studied the stagnation point flow. Moore (1951) analysed the problem of the laminar compressible boundary layer over an insulated flat plate moving with a time dependent velocity and, later, Ostrach (1955) extended Moore's results to the case of an isothermal flat plate.

This paper presents a solution of the problem of unsteady incompressible laminar boundary-layer flow for a certain class of outer pressure distributions. The method employed makes use of a transformation, a special case of which was given by the author in a recent note (Hassan 1960), which reduces the governing equation to an equation in which the time does not appear explicitly. In spite of the fact that such a scheme restricts to a great extent the free-stream velocity distributions that can be considered, yet the distribution considered includes, as special cases, a number of flows which are of interest.

It is assumed that the solution has a power series representation and it is shown that it can be recast in terms of universal functions. As an example of skin-friction calculations, the special case of the assumed flow where the free-stream velocity is a function of time is considered.

2. Basic equations and transformations

The differential equation for unsteady incompressible laminar boundary layer in two dimensions can be written as

$$\psi_{yt} + \psi_y \psi_{xy} - \psi_x \psi_{yy} = V_t + VV_x + \nu \psi_{yyy}, \tag{2.1}$$

where ψ is the stream function, V is the free-stream velocity and ν is the kinematic viscosity. The boundary conditions are

$$\psi(x,y,t) = \psi_y(x,y,t) = \psi_x(x,y,t) = 0 \quad \text{at} \quad y = 0,$$
$$\psi_y(x,y,t) \to V(x,t) \quad \text{as} \quad y \to \infty. \tag{2.2}$$

Letting $s = (x/l)\,(2\nu t/l^2)^{\frac{1}{2}(\lambda-1)}, \quad \sigma = y/\surd(2\nu t),$

$$V = (\nu/l)\,(2\nu t/l^2)^{-\frac{1}{2}(1+\lambda)}\,h(s), \quad \psi = \nu(2\nu t/l^2)^{-\frac{1}{2}\lambda}\,\chi(s,\sigma), \tag{2.3}$$

where l is a characteristic length, λ is any real or complex number,

and $$h(s) = s^\alpha \sum_0^\infty a_n s^n, \quad a_0 \neq 0, \quad \alpha \text{ an integer}; \tag{2.4}$$

and substituting into (2.1) and (2.2) one finds that

$$\chi_{\sigma\sigma\sigma} + (1-\lambda)\,s[\chi_{s\sigma} - h'] + (1+\lambda)\,[\chi_\sigma - h]$$
$$+ \sigma\chi_{\sigma\sigma} + \chi_s\chi_{\sigma\sigma} - \chi_\sigma\chi_{s\sigma} + hh' = 0, \tag{2.5}$$

and $$\chi = \chi_\sigma = \chi_s = 0 \quad \text{at} \quad \sigma = 0; \quad \chi_\sigma \to h(s) \quad \text{as} \quad \sigma \to \infty. \tag{2.6}$$

It is seen from (2.3) and (2.4) that the assumed expression for the free-stream velocity represents a very special class of $V(x,t)$. The assumed expression for V makes it necessary for ψ to assume the form given in (2.3) because $V = \psi_y$ at infinity.

Equation (2.6) shows that the boundary conditions depend on $h(s)$. However, a substitution which renders the boundary conditions independent of the free-stream velocity and paves the way for expressing the solution in terms of universal functions can be written as (Hassan 1960)

$$\xi = s, \quad \eta = \sigma f(s), \quad \chi = (h/f)\,\phi(\xi,\eta), \tag{2.7}$$

where $$f(\xi) = \xi^{\frac{1}{2}(\alpha-1)} \quad \text{for} \quad \alpha \leqslant 0; \quad f(\xi) = 1 \quad \text{for} \quad \alpha > 0. \tag{2.8}$$

Introducing (2.7) into (2.5) and (2.6), one obtains

$$\phi_{\eta\eta\eta} + (1/f^2)\,[1+\lambda+(1-\lambda)\,(\xi h'/h)]\,[\phi_\eta - 1]$$
$$+ (1/f^2)\,[1+(1-\lambda)\,(\xi f'/f)]\,\eta\phi_{\eta\eta} + (1-\lambda)\,(\xi/f^2)\,\phi_{\xi\eta}$$
$$+ (1/f)\,(h/f)'\,\phi\phi_{\eta\eta} + (h/f^2)\,[\phi_\xi\phi_{\eta\eta} - \phi_\eta\phi_{\xi\eta}]$$
$$+ (h'/f^2)\,(1-\phi_\eta^2) = 0, \tag{2.9}$$

and $$\phi = \phi_\eta = \phi_\xi = 0 \quad \text{at} \quad \eta = 0; \quad \phi_\eta \to 1 \quad \text{as} \quad \eta \to \infty. \tag{2.10}$$

3. Solution of equation (2.9)

The solution of (2.9) in the region of convergence of (2.4) will be assumed to be expressible as

$$\phi = \sum_{n=0}^\infty \phi_n(\eta)\,\xi^n. \tag{3.1}$$

Substituting (2.8) into (2.9) and letting

$$1+\lambda+(1-\lambda)(\xi h'/h) = \sum_{n=0}^{\infty} b_n \xi^n, \quad b_0 = 1+\alpha+\lambda(1-\alpha), \tag{3.2}$$

one obtains the following system of total differential equations for the functions ϕ_n.

(1) $\alpha \leqslant 0$:

$$\phi_0''' + \tfrac{1}{2}a_0(1+\alpha)\phi_0\phi_0'' + a_0\alpha(1-\phi_0'^2) = 0, \tag{3.3}$$

and

$$\phi_n''' + \tfrac{1}{2}a_0(1+\alpha)\phi_0\phi_n'' - a_0(n+2\alpha)\phi_0'\phi_n' + a_0[n+\tfrac{1}{2}(1+\alpha)]\phi_0''\phi_n = R_{n-1} \quad (n \geqslant 1), \tag{3.4}$$

where

$$\begin{aligned} R_{n-1} = Q_{n-1} \equiv a_n[(n+\alpha)(\phi_0'^2-1) - \{n+\tfrac{1}{2}(1+\alpha)\}\phi_0\phi_0''] \\ + a_0\sum_{k=1}^{n-1}[(k+\alpha)\phi_k'\phi_{n-k}' - \{k+\tfrac{1}{2}(1+\alpha)\}\phi_k\phi_{n-k}''] \\ - \phi_0\sum_{k=1}^{n-1}a_k[k+\tfrac{1}{2}(1+\alpha)]\phi_{n-k}'' + \phi_0'\sum_{k=1}^{n-1}a_k(k+\alpha)\phi_{n-k}' \\ + \sum_{k=1}^{n-1}\sum_{m=1}^{n-k}a_k[(m+k+\alpha)\phi_m'\phi_{n-m-k}' - \{m+k+\tfrac{1}{2}(1+\alpha)\}\phi_m\phi_{n-m-k}''], \\ \text{for} \quad n \leqslant -\alpha; \end{aligned} \tag{3.5a}$$

and

$$\begin{aligned} R_{n-1} = Q_{n-1} + b_{n+\alpha-1} - \sum_{k=0}^{n+\alpha-1} b_k\phi_{n+\alpha-k-1}' - \tfrac{1}{2}b_0\eta\phi_{n+\alpha-1}'' \\ -(1-\lambda)(n+\alpha-1)\phi_{n+\alpha-1}' \quad \text{for} \quad n > -\alpha. \end{aligned} \tag{3.5b}$$

It is to be understood that, in (3.5a), Q_{n-1} is identical to the right-hand side of the equation for all n, while R_{n-1} is equal to the right-hand side for restricted n as indicated.

(2) $\alpha = 1$:

$$\phi_0''' + 2(\phi_0'-1) + \eta\phi_0'' + a_0[1+\phi_0\phi_0''-\phi_0'^2] = 0, \tag{3.6}$$

and

$$\phi_n''' + (\eta + a_0\phi_0)\phi_n'' + [2+n(1-\lambda) - a_0(n+2)\phi_0']\phi_n' + a_0(n+1)\phi_0''\phi_n = R_{n-1} \quad (n \geqslant 1), \tag{3.7}$$

where

$$\begin{aligned} R_{n-1} = a_n(n+1)[\phi_0'^2 - \phi_0\phi_0'' - 1] + a_0\sum_{k=1}^{n-1}(k+1)[\phi_k'\phi_{n-k}' - \phi_k\phi_{n-k}''] \\ - \phi_0\sum_{k=1}^{n-1}a_k(k+1)\phi_{n-k}'' + \phi_0'\sum_{k=1}^{n-1}a_k(k+1)\phi_{n-k}' \\ + \sum_{k=1}^{n-1}\sum_{m=1}^{n-k}(m+k+1)a_k[\phi_m'\phi_{n-m-k}' - \phi_m\phi_{n-m-k}''] \\ - b_n(\phi_0'-1) - \sum_{k=1}^{n-1}b_k\phi_{n-k}'. \end{aligned} \tag{3.8}$$

(3) $\alpha > 1$:

$$\phi_0''' + \eta\phi_0'' + [1+\alpha+\lambda(1-\alpha)][\phi_0'-1] = 0, \tag{3.9}$$

and

$$\phi_n''' + \eta\phi_n'' + [1+\lambda+(n+\alpha)(1-\lambda)]\phi_n' = R_{n-1} \quad (n \geqslant 1), \tag{3.10}$$

where
$$R_{n-1} = Q_{n-1} \equiv -b_n(\phi_0' - 1) - \sum_{k=1}^{n-1} b_k \phi_{n-k}' \quad (n \leqslant \alpha - 2), \tag{3.11a}$$
and
$$R_{n-1} = Q_{n-1} + \sum_{k=0}^{n+1-\alpha} \sum_{m=0}^{n+1-\alpha-k} a_k(k+\alpha+n)\,[\phi_m' \phi_{n-m-k-\alpha+1}' - \phi_m \phi_{n-m-k-\alpha+1}''] - (n+1)\,a_{n+1-\alpha} \quad (n > \alpha - 2). \tag{3.11b}$$
The boundary conditions are
$$\phi_0(0) = \phi_0'(0) = 0, \quad \phi_0'(\infty) = 1; \quad \phi_n(0) = \phi_n'(0) = \phi_n'(\infty) = 0 \quad (n \geqslant 1). \tag{3.12}$$

The equations governing the zeroth order term, i.e. (3.3), (3.6) and (3.9), will be discussed next. Equation (3.3) can be easily reduced to the well-known equation of Falkner and Skan, and its solution was given by Hartree (1937). Equation (3.6), which is identical to the equation for unsteady stagnation-point flow, has been solved by Yang (1958). Finally, equation (3.9) is a linear equation which has a known solution:
$$\phi_0' = 1 - \eta^{-\frac{1}{2}} \exp(-\tfrac{1}{4}\eta^2) W_{k,m}(\tfrac{1}{2}\eta^2), \tag{3.13}$$
where $W_{k,m}$ is the confluent hypergeometric function (Whittaker & Watson 1927) with
$$k = \tfrac{1}{2}b_0 - \tfrac{1}{4}, \quad m = \tfrac{1}{4}. \tag{3.14}$$
Applying the boundary conditions reduces (3.13) to
$$\phi_0' = 1 - \exp(-\tfrac{1}{2}\eta^2)\,{}_1F_1[\tfrac{1}{2}(1-b_0), \tfrac{1}{2}, \tfrac{1}{2}\eta^2], \tag{3.15}$$
where
$${}_1F_1(a,b,x) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\ldots(a+n-1)}{b(b+1)\ldots(b+n-1)} \frac{x^n}{n!}. \tag{3.16}$$

The functions ϕ_n $(n \geqslant 1)$ are governed by linear equations and can be expressed as linear combinations of universal functions for a given α, λ and a_0. This can be accomplished by letting
$$\begin{aligned} \phi_1 &= \delta_1 f_1 + a_1 g_1 + \beta_1 h_1, \\ \phi_2 &= \delta_1^2 f_{11} + \delta_2 f_2 + a_1^2 g_{11} + a_2 g_2 + \beta_1^2 h_{11} + \beta_2 h_2 + \delta_1 a_1 p_{1,1} + a_1 \beta_1 q_{1,1} + \beta_1 \delta_1 r_{1,1}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{3.17}$$
where $\delta_n = 1$ for all n and $\beta_n = b_{n-1}$ for all $n \geqslant 1$. The governing equations for the universal functions f, g, ... are obtained by substituting the above equations into (3.4), (3.7) and (3.10) and equating coefficients of like constants. The boundary conditions satisfied by the universal functions are identical to those imposed on ϕ_n $(n \geqslant 1)$, namely, (3.2).

4. Skin-friction calculations

The wall shearing stress is given by
$$\tau_\omega = (\mu/2t)\,(2\nu t/l^2)^{-\frac{1}{2}\lambda}\, hf \sum_0^\infty \phi_n''(0)\,\xi^n, \tag{4.1}$$
and, the local skin-friction coefficient is
$$C_f = \tau_\omega/\tfrac{1}{2}\rho V^2 = (2f/h)\,(2\nu t/l^2)^{\frac{1}{2}\lambda} \sum_0^\infty \phi_n''(0)\,\xi^n. \tag{4.2}$$

As an example of skin-friction calculations, the simple case where $h(\xi) = a_0$ will be discussed. In this case,
$$V = a_0(\nu/l)\,(2\nu t/l^2)^{-\frac{1}{2}(1+\lambda)}, \tag{4.3}$$
$$f = 1/\sqrt{\xi}, \quad b_0 = 1 + \lambda, \quad b_n = 0 \quad (n \geqslant 1); \tag{4.4}$$

and the governing equations are

$$\phi_0''' + \tfrac{1}{2}a_0\phi_0\phi_n'' = 0, \tag{4.5}$$

$$\phi_1''' + \tfrac{1}{2}a_0\phi_0\phi_1'' - a_0\phi_0'\phi_1' + \tfrac{3}{2}a_0\phi_0''\phi_1 = b_0[1-\phi_0' - \tfrac{1}{2}\eta\phi_0''],$$
$$\cdots\cdots\cdots\cdots\cdots\cdots \tag{4.6}$$

Letting $$\phi_0 = a_0^{-\frac{1}{2}}F, \quad \phi_1 = -b_0 a_0^{-\frac{3}{2}}h_1, \quad \ldots, \quad \eta = 2a_0^{-\frac{1}{2}}\zeta, \tag{4.7}$$

we find that (4.5), (4.6) and (3.12) reduce to

$$F''' + FF'' = 0, \tag{4.8}$$

$$h_1''' + Fh_1'' - 2F'h_1' + 3F''h_1 = -4(2-F') + 2\zeta F'', \tag{4.9}$$

$$F(0) = F'(0) = 0, \quad F'(\infty) = 2, \quad h_1(0) = h_1'(0) = h_1'(\infty) = 0, \quad \ldots. \tag{4.10}$$

Equation (4.8) is the well-known Blasius equation, and (4.9) has been solved by Moore (1951). Therefore, one may write

$$C_f = [\tfrac{1}{2}a_0^{-\frac{1}{2}}\xi^{-\frac{1}{2}}](2\nu t/l^2)^{\frac{1}{2}\lambda}[F''(0) - (b_0/a_0)h_1''(0)\xi + \ldots]$$
$$= [\tfrac{1}{2}a_0^{-\frac{1}{2}}\xi^{-\frac{1}{2}}](2\nu t/l^2)^{\frac{1}{2}\lambda}[1{\cdot}328 - 3{\cdot}394\{(1+\lambda)/a_0\}\xi + \ldots]. \tag{4.11}$$

For accelerating flows $1+\lambda < 0$ and (4.11) shows that positive acceleration causes an increase in skin friction above the 'quasi-steady' value.

Considering the case of complex λ and letting

$$1+\lambda = a + i\omega, \tag{4.12}$$

one obtains

$$V = a_0(\nu/l)(2\nu t/l^2)^{-\frac{1}{2}a}\exp[-\tfrac{1}{2}i\omega\log(2\nu t/l^2)] \tag{4.13}$$

and $$C_f = \tfrac{1}{2}a_0^{-\frac{1}{2}}(l/x)^{\frac{1}{2}}(2\nu t/l^2)^{\frac{1}{4}a}\exp[\tfrac{1}{4}i\omega\log(2\nu t/l^2)][1{\cdot}328 - \ldots]. \tag{4.14}$$

Equations (4.13) and (4.14) show that the maximum of skin friction is not in phase with the maximum of the free-stream velocity.

5. Concluding remarks

The transformation that has been introduced makes it possible to give the solution of a wide class of unsteady flow problems without resorting to approximate methods. The method of solution gives a unified presentation to a number of special cases discussed in the literature and reduces the actual solution for a given problem to a solution of a few linear differential equations.

It should be noted, however, that although the method is well suited to problems in which $V(x,t) \sim x^n t^m$, it fails if $V(x,t) \sim f(x)F(t)$, where f and F are arbitrary functions.

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